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On a similar solution for a turbulent half-jet along a curved streamline

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An analysis of the mixing of a turbulent half-jet along a curved streamline is presented. Equations of motion referred to streamline co-ordinates are simplified by boundary-layer approximations and integrated under the assumption of similarity. A curved potential flow is dissipated by turbulence in the mixing region, resulting in a maximum value in the velocity distribution across the stream for convex flows, while the distribution is monotonic for concave flows. Pressure distributions across the stream are also presented.

1. Introduction

In many cases of turbulent mixing in boundary layers and free jets, the curvature of streamlines has no significant effect, but there are some cases, as shown in figure 1, in which the effect of streamline curvature should be taken into account, so that the pressure gradient across the streamline is no longer zero. Cases (*a*) and (*b*) are the boundary layer and the wall jet, respectively,

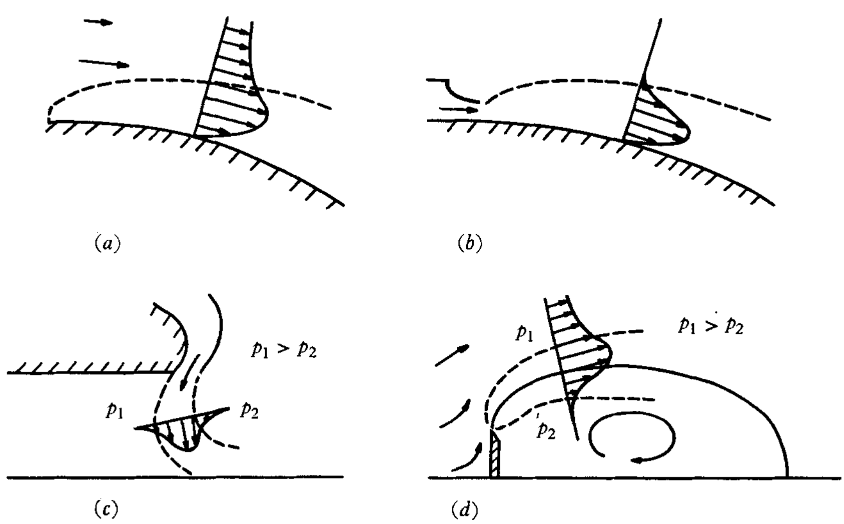

FIGURE 1. Flow patterns of curved mixing. (*a*) Curved boundary layer. (*b*) Wall jet (Coanda effect). (*c*) Curved jet. (*d*) Curved half-jet.

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along a curved solid surface. Cases (*c*) and (*d*) are the jet flows subject to a pressure difference. Case (*c*) is the curved jet as shown in ground-effect-machines and case (*d*) is the curved half-jet observed, for instance, in the re-circulating flow behind a flat plate placed perpendicularly in a uniform stream. The present analysis is concerned with the curved half-jet, in which the velocity is finite, on the upper side, while it is zero on the lower side. Uchida & Watanabe (1966, p. 677), and Uchida, Watanabe & Takada (1967) obtained a similar solution for laminar flow of such a curved half-jet referred to the streamline co-ordinates, which has been used by Yen & Toba (1961) for the curved boundary-layer flows. A similar procedure is extended to analyse turbulent mixing along a curved half-jet by replacing molecular kinematic viscosity by a turbulent kinematic viscosity which is proportional to the streamwise co-ordinate.

2. Fundamental equations

A steady two-dimensional flow of turbulent mixing along a curved streamline in incompressible fluid is considered. All physical variables are made non-dimensional in terms of some reference quantities, where $(1/2)\tilde{\rho}_s\tilde{u}_s^2$ is used as the

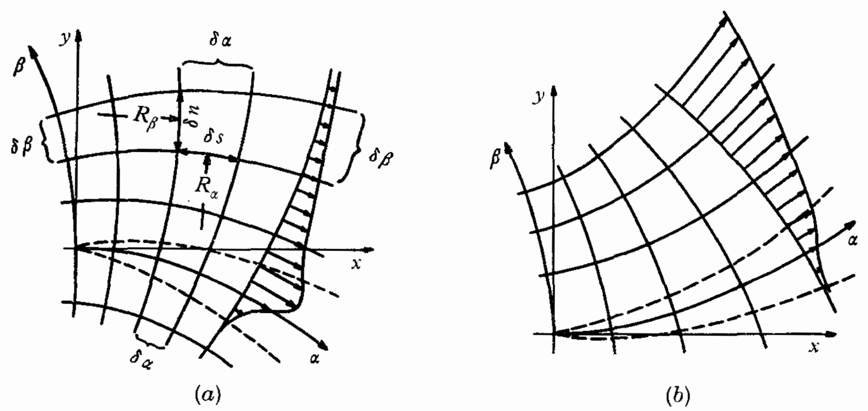

FIGURE 2. Streamline co-ordinates. (*a*) Convex flow ($C > 0$). (*b*) Concave flow ($C < 0$).

standard of stresses. Reynolds number is also defined by those reference quantities as $R_e = \tilde{u}_s\tilde{l}_s/\tilde{\nu}_s$.

In order to describe the flow in a curved half-jet, the streamline co-ordinates, α and β, are employed as shown in figure 2. The extension parameters of the co-ordinates are defined by $h_\alpha = \partial s/\partial\alpha$ and $h_\beta = \partial n/\partial\beta$, where s and n are the arc lengths along a streamline and along a normal to the streamline, respectively. These variables must satisfy the condition of orthogonality of the co-ordinates, which is expressed by Gauss's equation

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)=0. \tag{1}$$

To satisfy the equation of continuity

$$\operatorname{div}\mathbf{V}=0 \tag{2}$$

the stream function ψ is introduced, and is taken as equal to β, one of the streamline co-ordinates. Then the velocity component u is simply given by $1/h_\beta$ and v is identically equal to zero,

$$u \equiv \frac{1}{h_\beta}\frac{\partial\psi}{\partial\beta} = \frac{1}{h_\beta}, \quad v \equiv -\frac{1}{h_\alpha}\frac{\partial\psi}{\partial\alpha} = 0. \tag{3}$$

In order to derive the equation of motion in an explicit form, turbulent stresses are assumed to be expressed by the use of turbulent kinematic viscosity ϵ in place of molecular kinematic viscosity ν. In general cases ϵ should be a function of place, i.e. of α and β. This treatment seems to be most convenient to analyse such turbulent flows deformed in a two-dimensional field.

Normal stresses $\sigma_{\alpha\alpha}$ and $\sigma_{\beta\beta}$, and shear stress $\tau_{\alpha\beta}$ which are made non-dimensional in term of reference dynamic pressure $(\frac{1}{2})\tilde{\rho}_s\tilde{u}_s^2$, are given in the general orthogonal curvilinear co-ordinates as follows:

$$\sigma_{\alpha\alpha} \equiv -p+\tau_{\alpha\alpha}, \quad \sigma_{\beta\beta} \equiv -p+\tau_{\beta\beta}, \tag{4}$$

where
$$\tau_{\alpha\alpha} = \frac{2}{R_e}\left\{2\rho\epsilon\left[\frac{1}{h_\alpha}\frac{\partial u}{\partial\alpha}+\frac{v}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right]-\frac{2}{3}\rho\epsilon\,\mathrm{div}\,\mathbf{V}\right\},$$
$$\tau_{\beta\beta} = \frac{2}{R_e}\left\{2\rho\epsilon\left[\frac{1}{h_\beta}\frac{\partial v}{\partial\beta}+\frac{u}{h_\beta h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right]-\frac{2}{3}\rho\epsilon\,\mathrm{div}\,\mathbf{V}\right\} \tag{5}$$

and
$$\tau_{\alpha\beta} = \tau_{\beta\alpha} = \frac{2}{R_e}\left\{\rho\epsilon\left[\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{v}{h_\beta}\right)+\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right]\right\}. \tag{6}$$

ρ and ϵ are made non-dimensional in terms of reference density $\tilde{\rho}_\alpha$ and reference kinematic viscosity $\tilde{\nu}_s$, respectively. ρ is put unity hereafter. The force components in α and β directions acting on a fluid element of unit volume are expressed by

$$F_\alpha = -\frac{1}{h_\alpha}\frac{\partial p}{\partial\alpha}+\frac{1}{h_\alpha}\frac{\partial\tau_{\alpha\alpha}}{\partial\alpha}+\frac{1}{h_\beta}\frac{\partial\tau_{\beta\alpha}}{\partial\beta}+\frac{2}{h_\alpha h_\beta}\tau_{\alpha\beta}\frac{\partial h_\alpha}{\partial\beta}+\frac{1}{h_\alpha h_\beta}(\tau_{\alpha\alpha}-\tau_{\beta\beta})\frac{\partial h_\beta}{\partial\alpha} \tag{7}$$

and F_β which is obtained by the cyclic change of variables. It is noted here that F_α and F_β are made non-dimensional by referring to $(\frac{1}{2})\tilde{\rho}_s\tilde{u}_s^2/\tilde{l}_s$.

The equations of motion referred to the orthogonal curvilinear co-ordinates are obtained from (5), (6), (7) and (2). They are given by

$$\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{u^2+v^2}{2}\right)-v\zeta = -\frac{1}{2h_\alpha}\frac{\partial p}{\partial\alpha}-\frac{\epsilon}{R_e}\frac{1}{h_\beta}\frac{\partial\zeta}{\partial\beta}+\frac{1}{R_e}\frac{1}{h_\alpha}\frac{\partial\epsilon}{\partial\alpha}\left[\frac{2}{h_\alpha}\frac{\partial u}{\partial\alpha}+\frac{2v}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right]$$
$$+\frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial\epsilon}{\partial\beta}\left[\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{v}{h_\beta}\right)+\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right] \tag{8}$$

for α component and its cyclic change of variables for β component, where ζ denotes the vorticity defined by

$$\zeta = (1/h_\alpha h_\beta)\left[\partial(h_\beta v)/\partial\alpha-\partial(h_\alpha u)/\partial\beta\right]. \tag{9}$$

The equations of motion referred to the streamline co-ordinates are given by putting $v = 0$ in (8). The α and β components of the equations are:

$$\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(\tfrac{1}{2}u^2) = -\frac{1}{2h_\alpha}\frac{\partial p}{\partial\alpha}-\frac{\epsilon}{R_e}\frac{1}{h_\beta}\frac{\partial\zeta}{\partial\beta}+\frac{1}{R_e}\frac{1}{h_\alpha}\frac{\partial\epsilon}{\partial\alpha}\frac{2}{h_\alpha}\frac{\partial u}{\partial\alpha}+\frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial\epsilon}{\partial\beta}\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right), \tag{10}$$

$$\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left(\tfrac{1}{2}u^2\right)+u\zeta=-\frac{1}{2h_\beta}\frac{\partial p}{\partial\beta}+\frac{\epsilon}{R_e}\frac{1}{h_\alpha}\frac{\partial\zeta}{\partial\alpha}$$
$$+\frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial\epsilon}{\partial\beta}\frac{2u}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\alpha}+\frac{1}{R_e}\frac{1}{h_\alpha}\frac{\partial\epsilon}{\partial\alpha}\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right), \tag{11}$$

where
$$\zeta=-(1/h_\alpha h_\beta)\left[\partial(h_\alpha u)/\partial\beta\right]. \tag{12}$$

3. Boundary-layer approximation

The boundary-layer approximation is now introduced, in which the orders of magnitude are assumed to be

$$\text{Order}\,[\beta]=R_e^{-\frac{1}{2}},\ \text{Order}\,[h_\alpha,h_\beta,u,p,\alpha,\epsilon]=1.$$

Gauss's equation of orthogonality and equations of motion are then simplified into the following equations:

$$\frac{\partial}{\partial\beta}\left(\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)=0, \tag{13}$$

$$\frac{u}{h_\alpha}\frac{\partial u}{\partial\alpha}=-\frac{1}{2h_\alpha}\frac{\partial p}{\partial\alpha}+\frac{\epsilon}{R_e}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial(h_\alpha u)}{\partial\beta}\right]+\frac{1}{R_e}\frac{1}{h_\beta}\frac{\partial\epsilon}{\partial\beta}\left[\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right], \tag{14}$$

$$-\frac{u^2}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta}=-\frac{1}{2h_\beta}\frac{\partial p}{\partial\beta},\quad\text{or}\quad-\frac{u^2}{R_\alpha}=-\frac{1}{2h_\beta}\frac{\partial p}{\partial\beta}, \tag{15}$$

where $1/R_\alpha=(1/h_\alpha h_\beta)\left[\partial h_\alpha/\partial\beta\right]$ represents curvature of streamlines. Equation (15) expresses that the centrifugal force is counter balanced by pressure gradient across the stream. It is found that the effect of curvature of streamlines should be taken into account when its magnitude has the order of $R_e^{\frac{1}{2}}$. Elimination of pressure from the equations of motion yields the vorticity equation

$$\frac{\partial}{\partial\alpha}(h_\beta u\zeta)=\frac{1}{R_e}\frac{\partial}{\partial\beta}\left(\epsilon\frac{h_\alpha}{h_\beta}\frac{\partial\zeta}{\partial\beta}\right)-\frac{1}{R_e}\frac{\partial}{\partial\beta}\left[\frac{h_\alpha^2}{h_\beta^2}\frac{\partial\epsilon}{\partial\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right], \tag{16}$$

where $\zeta=-(1/h_\alpha h_\beta)\left[\partial(h_\alpha u)/\partial\beta\right]$, and $h_\beta u=1$ from equation (3) should be substituted. The problem is now to solve h_α and h_β as functions of α and β from the equations (13) and (16). Equation (15) will be used to determine pressure distribution.

As the dependent variables, h_α and $\lambda\equiv h_\alpha/h_\beta$ are used rather than h_α and h_β. With these variables, Gauss's equation and the vorticity equation are given by

$$\frac{\partial}{\partial\beta}\left(\frac{\lambda}{h_\alpha}\frac{\partial h_\alpha}{\partial\beta}\right)=0, \tag{17}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial\lambda}{\partial\beta}\right)=\frac{1}{R_e}\frac{\partial}{\partial\beta}\left[\epsilon\lambda\frac{\partial}{\partial\beta}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial\lambda}{\partial\beta}\right)\right]+\frac{1}{R_e}\frac{\partial}{\partial\beta}\left[\frac{\partial\epsilon}{\partial\beta}\lambda^2\frac{\partial}{\partial\beta}\left(\frac{\lambda}{h_\alpha^2}\right)\right]. \tag{18}$$

Boundary conditions are required for the flow to be smoothly joined to outer non-viscous potential flow at $\beta=+\infty$ and to zero velocity at $\beta=-\infty$. Since $\lambda=u/(1/h_\alpha)$ and $h_\alpha=h_\beta$ in potential flow, λ should tend to unity at $\beta=+\infty$ with vanishing derivative $\partial\lambda/\partial\beta$, and λ should tend to zero at $\beta=-\infty$. These boundary conditions are expressed by

$$\left.\begin{array}{llll}\lambda = 1, & \partial\lambda/\partial\beta = 0 & \text{at} & \beta = +\infty, \\ \lambda = 0, & & \text{at} & \beta = -\infty.\end{array}\right\} \tag{19}$$

It is seen that λ is the velocity ratio, or the reduction of velocity in the mixing region due to turbulent stresses, while $1/h_\alpha$ may be interpreted as the velocity of a hypothetical non-viscous flow, since it is equal to the velocity u at $\beta = +\infty$.

4. Similar solutions

It is found that a similar solution can be obtained by introducing the single independent variable η and its functions:

$$\left.\begin{array}{c}\eta = (R_e/\sigma)^{\frac{1}{2}}.\beta/\alpha, \quad \lambda = \Lambda(\eta), \quad h_\alpha = \alpha^m H(\eta), \quad \epsilon = 2\sigma\alpha, \\ u = 1/h_\beta = \alpha^{-m}\Lambda/H, \quad p = \alpha^{-2m}P(\eta),\end{array}\right\} \tag{20}$$

where m represents the effects of streamwise pressure gradient. It is seen that the width of mixing region develops in proportion to α.

Gauss's equation is transformed and integrated to the form

$$[(\Lambda/H)\,H']' = 0, \quad \text{giving} \quad (\Lambda/H)\,H' = C \tag{21}$$

where $'$ means $d/d\eta$ and the constant C represents a curvature parameter as shown later.

The vorticity equation is given by

$$2[\Lambda(H^{-2}\Lambda\Lambda')']' + \eta(H^{-2}\Lambda\Lambda')' + (1+2m)\,H^{-2}\Lambda\Lambda' = 0. \tag{22}$$

By the use of (21), H is eliminated from the vorticity equation, which then gives the differential equation for Λ

$$2\Lambda^2(\Lambda^2)''' + [(\Lambda^2)' + (\eta - 8C)\,\Lambda]\,(\Lambda^2)'' + [(1+2m)\,\Lambda - 2C(\eta - 4C)]\,(\Lambda^2)' = 0. \tag{23}$$

Boundary conditions for Λ are

$$\left.\begin{array}{llll}\Lambda = 1, & \Lambda' = 0, & \text{at} & \eta = +\infty, \\ \Lambda = 0, & & \text{at} & \eta = -\infty.\end{array}\right\} \tag{24}$$

H can be integrated from (21). Assuming $H = H_0$ at $\eta = 0$, it is given as a function of Λ:

$$H/H_0 = \exp\left[C\int_0^\eta \Lambda^{-1}\,d\eta\right]. \tag{25}$$

When Λ and H are obtained, the velocity u can be calculated by

$$u = 1/h_\beta = \lambda/h_\alpha = H_0^{-1}\alpha^{-m}[\Lambda(H/H_0)^{-1}]. \tag{26}$$

Substituting (20) into (15), the pressure gradient is expressed by functions of the similarity variable:

$$P' = 2(\Lambda^2/H^3)\,H', \quad \text{or} \quad P' = 2C(\Lambda/H^2). \tag{27}$$

It is integrated to the form

$$P = P_0 + 2H_0^{-2}C\int_0^\eta \Lambda(H/H_0)^{-2}\,d\eta. \tag{28}$$

The shearing stress along zero-streamline is calculated from (6). Since the non-dimensional density ρ equals unity in incompressible flow, it is

$$(\tau_{\alpha\beta})_{\beta=0} = \frac{2}{R_e}\epsilon\left[\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{u}{h_\alpha}\right)\right]_{\beta=0}. \tag{29}$$

Substituting (20), it is given by

$$\begin{aligned}(\tau_{\alpha\beta})_{\beta=0} &= 4H_0^{-2}(R_e/\sigma)^{-\frac{1}{2}}\alpha^{-2m}[\Lambda(\Lambda H^{-2}H_0^2)']_{\eta=0}\\ &= 4H_0^{-2}(R_e/\sigma)^{-\frac{1}{2}}\alpha^{-2m}[\Lambda(\Lambda'-2C)]_{\eta=0}.\end{aligned} \tag{30}$$

The form of zero-streamline is now calculated by transforming the curvature of streamlines

$$1/R_\alpha = (1/h_\alpha h_\beta)[\partial h_\alpha/\partial\beta] = (R_e/\sigma)^{\frac{1}{2}}\alpha^{-(1+m)}\Lambda H^{-2}H'. \tag{31}$$

Substituting (21) and $H = H_0$ along the zero-streamline, we have

$$(1/R_\alpha)_{\beta=0} = C(R_e/\sigma)^{\frac{1}{2}}\alpha^{-(1+m)}/H_0. \tag{32}$$

It is noted here that the constant C represents a curvature parameter, positive for the flow along a convex streamline, and negative for a concave streamline. Denoting the angle of inclination of zero-streamline by θ, its curvature is calculated by

$$(1/R_\alpha)_{\beta=0} = -\partial\theta/\partial s = -(1/h_\alpha)[\partial\theta/\partial\alpha]. \tag{33}$$

A relation between θ and α along the zero-streamline is obtained from (32) and (33)

$$d\theta = -C(R_e/\sigma)^{\frac{1}{2}}\alpha^{-1}d\alpha, \tag{34}$$

which is integrated to

$$\alpha = \exp[-C^{-1}(R_e/\sigma)^{-\frac{1}{2}}(\theta-\theta_0)]. \tag{35}$$

The derivative $d\alpha/d\theta$ is given by

$$\frac{d\alpha}{d\theta} = -\frac{1}{C(R_e/\sigma)^{\frac{1}{2}}}\exp\left[-\frac{\theta-\theta_0}{C(R_e/\sigma)^{\frac{1}{2}}}\right]. \tag{36}$$

In order to calculate the form in cartesian co-ordinates, $\partial x/\partial s = \cos\theta$ and $\partial y/\partial s = \sin\theta$ are transformed by

$$(\partial s/\partial\alpha)_{\beta=0} = (h_\alpha)_{\beta=0} = \alpha^m H_0.$$

Those are given

$$\partial x/\partial\alpha = H_0\alpha^m\cos\theta,\ \partial y/\partial\alpha = H_0\alpha^m\sin\theta. \tag{37}$$

Substituting (35) and (36) into (37), x and y co-ordinates of zero-streamline are calculated by

$$\left.\begin{aligned}x &= x_0 - \frac{H_0}{C(R_e/\sigma)^{\frac{1}{2}}}\int_0^\theta \exp\left[-\frac{1+m}{C(R_e/\sigma)^{\frac{1}{2}}}(\theta-\theta_0)\right]\cos\theta\,d\theta,\\ y &= y_0 - \frac{H_0}{C(R_e/\sigma)^{\frac{1}{2}}}\int_0^\theta \exp\left[-\frac{1+m}{C(R_e/\sigma)^{\frac{1}{2}}}(\theta-\theta_0)\right]\sin\theta\,d\theta.\end{aligned}\right\} \tag{38}$$

Integrated forms of x and y are given with a parameter

$$a = (1+m)^{-1}C(R_e/\sigma)^{\frac{1}{2}}$$

as follows:

$$\left.\begin{aligned}x &= x_0 + \frac{H_0 e^{\theta_0/a}}{1+m}\frac{(\cos\theta - a\sin\theta)e^{-\theta/a}-1}{1+a^2},\\ y &= y_0 + \frac{H_0 e^{\theta_0/a}}{1+m}\frac{(\sin\theta + a\cos\theta)e^{-\theta/a}-a}{1+a^2}.\end{aligned}\right\} \tag{39}$$

The non-dimensional arc length along streamlines is calculated by

$$s = \int h_\alpha \, d\alpha = s_0 - (R_e/\sigma)^{\frac{1}{2}(1+m)} \beta^{1+m} \int_0^\eta H \eta^{-(2+m)} \, d\eta.$$

Along zero-streamline it is integrated to

$$(s)_{\beta=0} = s_0 + H_0 \alpha^{1+m}/(1+m). \tag{40}$$

The arc length normal to the streamline is given by

$$\begin{aligned} n = \int h_\beta \, d\beta &= n_0 + \alpha^{1+m} (R_e/\sigma)^{-\frac{1}{2}} \int_0^\eta (H/\Lambda) \, d\eta \\ &= n_0 + H_0 \alpha^{1+m} (R_e/\sigma)^{-\frac{1}{2}} C^{-1} [(H/H_0) - 1]. \end{aligned} \tag{41}$$

A combined parameter $(R_e/\sigma)^{\frac{1}{2}} n/s$, which corresponds to $\eta = (R_e/\sigma)^{\frac{1}{2}} \beta/\alpha$, may be used conveniently to express distributions of quantities in similar solution. Choosing $s_0 = n_0 = 0$ it is given by

$$(R_e/\sigma)^{\frac{1}{2}} n/s = (1+m) C^{-1} [(H/H_0) - 1]. \tag{42}$$

5. Potential flows matching to the curved half-jet

The outer potential flow matching to the inner mixing zone of curved half-jet is now considered. Fundamental equations for potential flow are

$$\operatorname{div} \mathbf{V} = 0 \quad \text{and} \quad \zeta = 0 \tag{43}$$

and are identically satisfied by a stream function ψ and velocity potential ϕ, which are defined by

$$\left. \begin{aligned} U &= (1/h_\beta)(\partial\psi/\partial\beta) = (1/h_\alpha)(\partial\phi/\partial\alpha), \\ V &= -(1/h_\alpha)(\partial\psi/\partial\alpha) = (1/h_\beta)(\partial\phi/\partial\beta). \end{aligned} \right\} \tag{44}$$

In order to refer to streamline co-ordinates, α and β are taken to be

$$\alpha = \phi, \quad \beta = \psi. \tag{45}$$

Then the velocity components are

$$U = 1/h_\beta = 1/h_\alpha, \quad V = 0. \tag{46}$$

It is known that the flow pattern becomes homogeneous as shown by $h_\alpha = h_\beta$. Substituting

$$h_\alpha = h_\beta = h$$

into Gauss's equation, we have

$$\partial^2(\ln h)/\partial\alpha^2 + \partial^2(\ln h)/\partial\beta^2 = 0. \tag{47}$$

The solution should be matched along $\beta = 0$ with the half-jet solution at $\beta \to +\infty$. If we denote the value of H at $\eta \to +\infty$, which tends to ∞ or 0 in the convex or concave case respectively, temporarily by H_∞, h and its derivatives at $\beta = 0$ are

$$\left. \begin{aligned} (h)_{\beta=0} &= H_\infty \alpha^m, \\ (\partial \ln h/\partial\beta)_{\beta=0} &= (h/R_\alpha)_{\beta=0} = C(R_e/\sigma)^{\frac{1}{2}} \alpha^{-1}, \\ (\partial^2 \ln h/\partial\beta^2)_{\beta=0} &= (-\partial^2 \ln h/\partial\alpha^2)_{\beta=0} = -m\alpha^{-2}. \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \right\} \tag{48}$$

The complete form of $\ln h$ can be calculated by constructing a series expansion from (48) as follows:

$$\begin{aligned}\ln h &= \ln H_\infty + m\ln\alpha + C\left(\frac{R_e}{\sigma}\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\left(\frac{\beta}{\alpha}\right)^{2j+1} + \frac{m}{2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{\beta}{\alpha}\right)^{2j} \\ &= \ln H_\infty + \ln\alpha^m + C(R_e/\sigma)^{\frac{1}{2}}\tan^{-1}(\beta/\alpha) + (\tfrac{1}{2}m)\ln[1+(\beta/\alpha)^2], \end{aligned} \tag{49}$$

or

$$h = H_\infty \alpha^m\left[1+\left(\frac{\beta}{\alpha}\right)^2\right]^{\frac{1}{2}m}\exp\left[C\left(\frac{R_e}{\sigma}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right]. \tag{50}$$

The velocity of potential flow is

$$U = \frac{1}{h} = \frac{1}{H_\infty}\alpha^{-m}\left[1+\left(\frac{\beta}{\alpha}\right)^2\right]^{-\frac{1}{2}m}\exp\left[-C\left(\frac{R_e}{\sigma}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right]. \tag{51}$$

The velocity along zero-streamline is

$$U_0 = 1/h_0 = (1/H_\infty)\,\alpha^{-m} = (1/h_\alpha)_{\beta\to\infty}. \tag{52}$$

Then the velocity distribution of potential flow based on the velocity on the zero-streamline is given as

$$\frac{U}{U_0} = \frac{h_0}{h} = \left[1+\left(\frac{\beta}{\alpha}\right)^2\right]^{-\frac{1}{2}m}\exp\left[-C\left(\frac{R_e}{\sigma}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right]. \tag{53}$$

In the inner region of $\beta \doteqdot 0$, it tends to

$$\frac{U}{U_0} = \frac{h_0}{h} \doteqdot \left[1-\frac{m}{2}\left(\frac{\beta}{\alpha}\right)^2\right]\exp\left[-C\left(\frac{R_e}{\sigma}\right)^{\frac{1}{2}}\frac{\beta}{\alpha}\right] = \left[1-\frac{m}{2}\left(\frac{\beta}{\alpha}\right)^2\right]e^{-C\eta}. \tag{54}$$

6. Similar solutions with zero pressure gradient along the stream

The solution of zero streamwise pressure gradient, which is given by $m = 0$, will be the most significant case of similar solutions. Similarity conditions for this particular case are

$$\left.\begin{aligned}&\eta = (R_e/\sigma)^{\frac{1}{2}}\beta/\alpha, \quad \lambda = \Lambda(\eta), \quad h_\alpha = H(\eta), \quad \epsilon = 2\sigma\alpha, \\ &u = 1/h_\beta = \Lambda/H, \quad p = P(\eta).\end{aligned}\right\} \tag{55}$$

Elimination of the variable H from Gauss's equation (21) and the vorticity equation (22) leads to an ordinary differential equation for Λ

$$[2\Lambda(\Lambda^2)'' + (\eta-4C)(\Lambda^2)']' - 2C\Lambda^{-1}[2\Lambda(\Lambda^2)'' + (\eta-4C)(\Lambda^2)'] = 0, \tag{56}$$

which can be integrated once to give

$$2\Lambda(\Lambda^2)'' + (\eta-4C)(\Lambda^2)' = B\exp\left[2C\int_0^\eta \Lambda^{-1}d\eta\right]. \tag{57}$$

With non-zero value of B, the right-hand side of equation (57) tends to $+\infty$ at $\eta = +\infty$ for $C > 0$, and to $-\infty$ at $\eta = -\infty$ for $C < 0$. In order to avoid this singularity B is set to zero and we have

$$\Lambda\Lambda'' + (\Lambda')^2 + (\tfrac{1}{2})(\eta-4C)\Lambda' = 0. \tag{58}$$

Boundary conditions for Λ are

$$\left.\begin{array}{llll}\Lambda = 1, & \Lambda' = 0 & \text{at} & \eta = +\infty, \\ \Lambda = 0 & & \text{at} & \eta = -\infty,\end{array}\right\} \tag{59}$$

in which the condition $\Lambda' = 0$ is automatically satisfied by putting $B = 0$.

Equation (58) is integrated numerically with boundary conditions (59), by connecting with the asymptotic expansion of Λ in the outer region

$$\Lambda_{\eta\to\infty} = 1 - \sqrt{\pi} A \operatorname{erfc}[(\eta - 4C)/2] \tag{60}$$

at $\eta - 4C = 7$. It is found that $\Lambda(-\infty) = \text{constant} = K$ for $A < A_{cr}$ and $\Lambda(-\infty) \to -\infty$ for $A > A_{cr}$, where A_{cr} is a critical value of the constant A. The present boundary condition at $\eta = -\infty$ is satisfied by the limit solution given by $K \to 0$, resulting in a value of $A = 0{\cdot}16468$.

The function H is calculated by

$$H_0/H = \exp\left[-C\int_0^{\eta} \Lambda^{-1}\,d\eta\right], \tag{61}$$

where H_0 is the value of H at $\eta = 0$. In the outer region of $\eta > \eta_M$, Λ is very close to unity, and therefore, $1/H$ tends to $e^{-C\eta}$. In this region H_0/H tends to

$$H_0/H = (H_0/H_M)(H_M/H) \doteqdot (H_0/H_M)\,e^{-C(\eta-\eta_M)}, \tag{62}$$

where H_M is the value of H at the matching point η_M.

In the inner region of outer potential flow the velocity distribution is given by equation (54),

$$U/U_0 = h_0/h = e^{-C\eta} \tag{63}$$

or

$$U/U_0 = (h_0/h_M)(h_M/h) = (h_0/h_M)\,e^{-C(\eta-\eta_M)} \tag{64}$$

where h_M is the value of h at $\eta = \eta_M$.

It is found that the velocity in the outer region of mixing layer can be smoothly connected with the velocity of potential flow by putting

$$H_M = h_M \quad \text{at} \quad \eta = \eta_M. \tag{65}$$

The matching is verified by

$$\frac{u}{U_0} \to \frac{H^{-1}}{U_0} = \frac{H_M^{-1}}{U_0}\,e^{-C(\eta-\eta_M)} = \frac{h_M^{-1}}{h_0^{-1}}\,e^{-C(\eta-\eta_M)} = \frac{U}{U_0}. \tag{66}$$

In the present calculation the matching point is chosen at $\eta_M = 6 + 4C$.

The results of numerical integration for several values of C are shown in figure 3, positive value of C corresponding to the flow along a convex streamline. Solid lines show the values of velocity ratio. Λ, and broken lines the hypothetical velocity distribution H^{-1}/H_0^{-1}.

The distribution of velocity along the normal to streamline is calculated by $u/U_0 = \Lambda(H^{-1}/H_0^{-1})(H_0^{-1}/U_0)$ matching H^{-1} with the outer potential flow at $\eta_M = 6 + 4C$. Numerical examples are shown in figure 4. Solid lines show the values of flow velocity u/U_0 in the mixing layer, and chain lines the values of outer potential flow U/U_0.

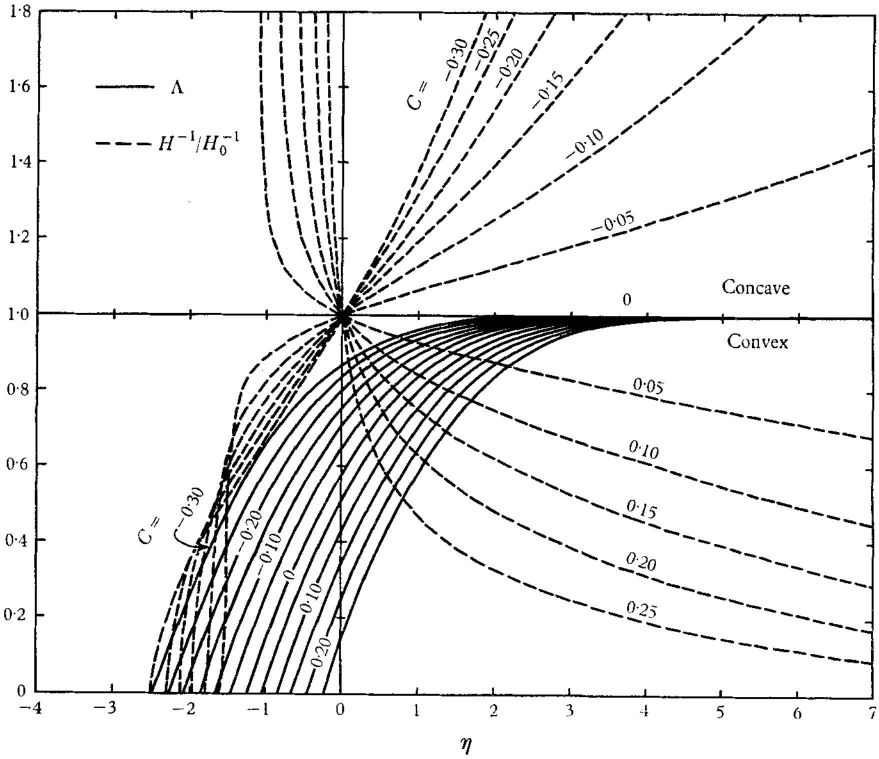

FIGURE 3. Similar solution, Λ and H^{-1}/H_0^{-1}.

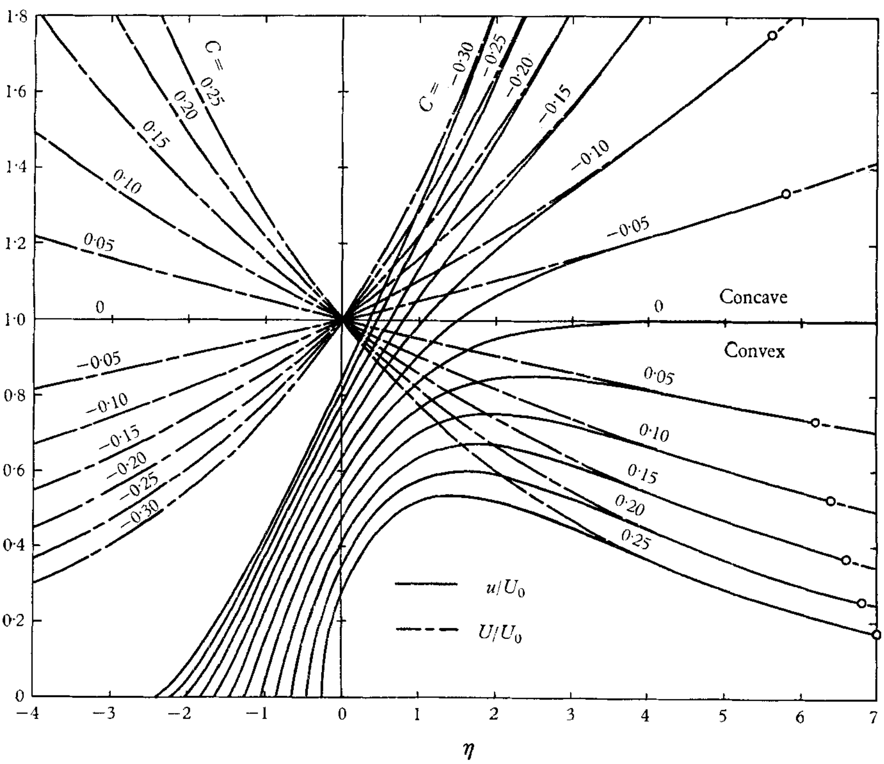

FIGURE 4. Similar solution, u/U_0 and U/U_0.

The form of zero-streamline which starts at the origin with zero initial slope is given by

$$\left.\begin{aligned} x &= H_0 e^{\theta_0/a}(1+a^2)^{-1}[(\cos\theta - a\sin\theta)\, e^{-\theta/a} - 1], \\ y &= H_0 e^{\theta_0/a}(1+a^2)^{-1}[(\sin\theta + a\cos\theta)\, e^{-\theta/a} - a], \end{aligned}\right\} \quad (67)$$

where a is a combined curvature parameter defined by $a = C(R_e/\sigma)^{\frac{1}{2}}$. Several examples are shown in figure 5, where convex streamlines for $a > 0$ or $C > 0$

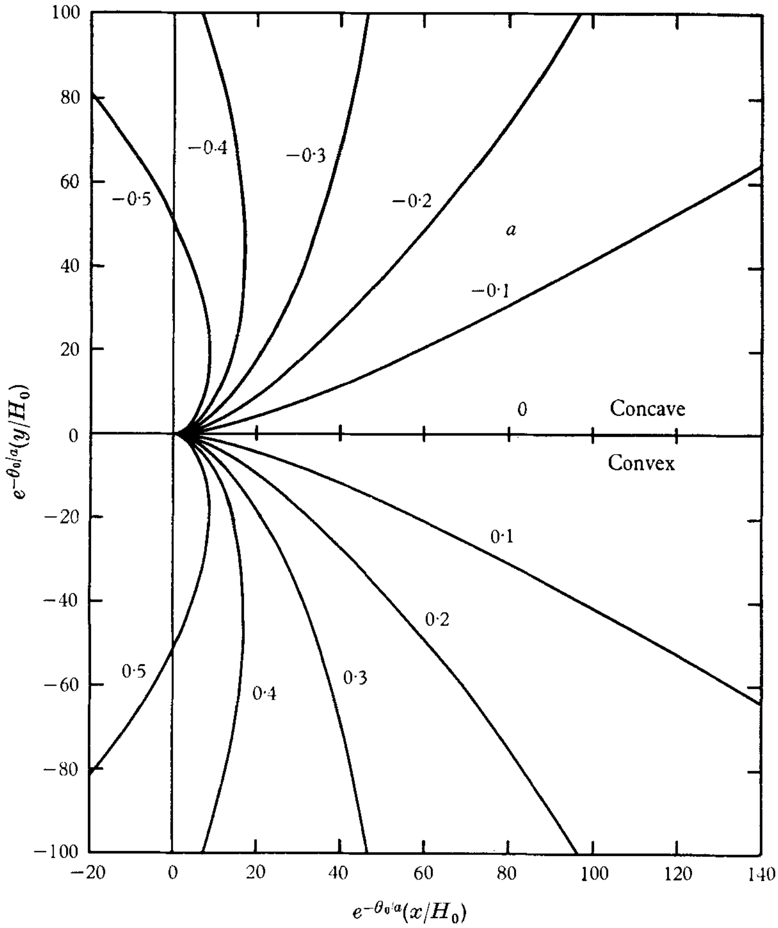

FIGURE 5. Zero-streamlines.

bend downward while those $a < 0$ or $C < 0$ bend upward. In these cases the non-viscous flow is considered to be far above the zero-streamlines.

The non-dimensional arc length of zero-streamline starting at the origin and that of normal lines to the stream originating at the zero-streamline are given from equations (40) and (41),

$$s = H_0\alpha, \quad n = H_0\alpha(R_e/\sigma)^{-\frac{1}{2}}C^{-1}[(H/H_0)-1]. \quad (68)$$

A physical interpretation of the similarity parameter η is now introduced by

$$(R_e/\sigma)^{\frac{1}{2}}\, n/s = (1/C)\,[(H/H_0)-1]. \quad (69)$$

It is found that the similar solution can be expressed by this parameter in place of η. Equation (69) is transformed to give the velocity of hypothetical basic flow

$$H_0/H = [1+C(R_e/\sigma)^{\frac{1}{2}}\,n/s]^{-1}. \qquad (70)$$

The distribution of flow velocity expressed as a function of $(Re/\sigma)^{\frac{1}{2}}n/s$ is calculated and some examples are shown in figures 6, 7 and 8. For convex streamlines, the velocity of outer potential flow decreases as n increases (shown by

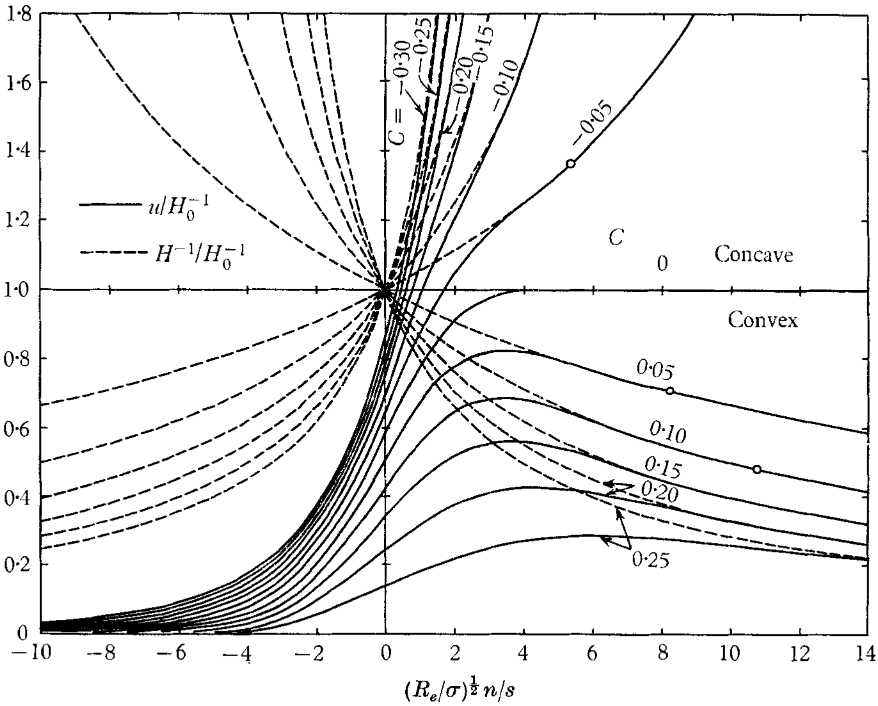

FIGURE 6. Velocity distributions, u/H_0^{-1}.

chain lines). The effect of turbulent mixing makes the flow decelerate in the inner part, resulting in the occurrence of a maximum velocity in the intermediate region. For concave flow no maximum velocity takes place, the velocity decreasing monotonically from outer to inner region.

In figure 8, the velocity is expressed in terms of the value of the zero-streamline. In this form of representation, the effect of increasing the parameter C from zero is somewhat complicated, at first decreasing, and then increasing. Decreasing C from zero makes the change only monotonic. The velocity distribution u/U_0 in figure 7 has also a somewhat similar character.

The pressure distribution across the streamlines is calculated by the integration of (27) originating from the equation of motion along the normal to streamlines. In this case the pressure distribution in the mixing layer is given from equation (28)

$$(p-p_0)/U_0^2 = (P-P_0)/U_0^2 = 2C(H_0^{-2}/U_0^2)\int_0^\eta \Lambda(H/H_0)^{-2}\,d\eta. \qquad (71)$$

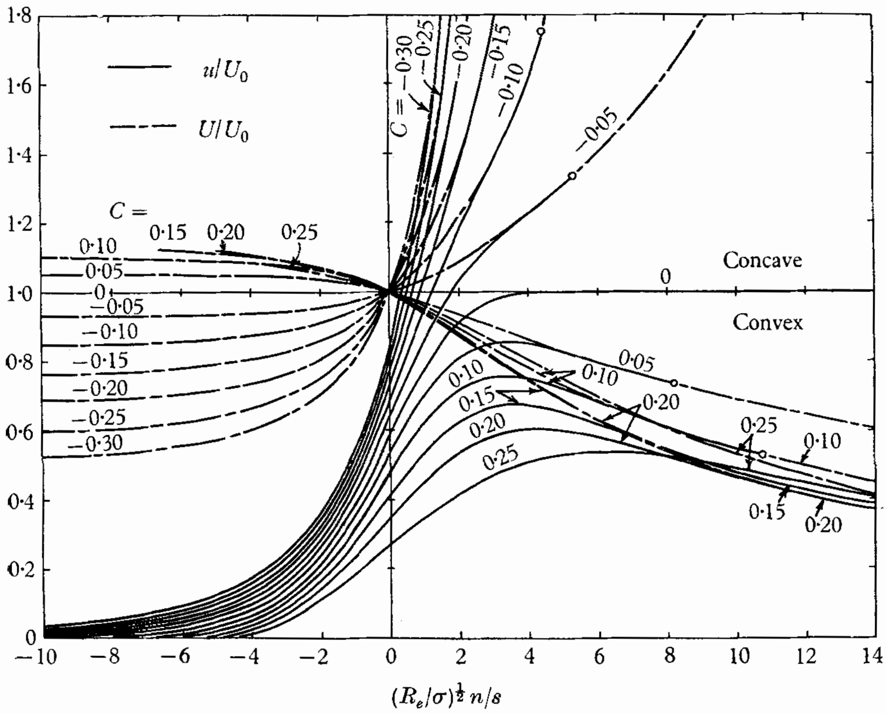

FIGURE 7. Velocity distributions, u/U_0.

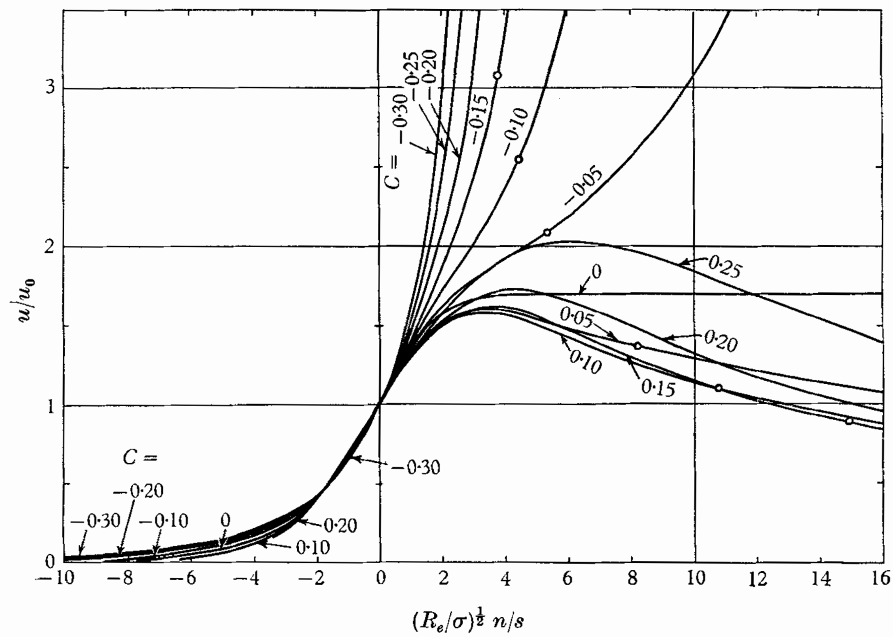

FIGURE 8. Velocity distributions, u/u_0.

In the region of outer potential flow at $\eta > \eta_M$, it is

$$\frac{p-p_0}{U_0^2} = \frac{p_M-p_0}{U_0^2} + \frac{p-p_M}{U_0^2} = 2C\frac{H_0^{-2}}{U_0^2}\int_0^{\eta} \Lambda \left(\frac{H_0}{H}\right)^2 d\eta + \left(\frac{U_M^2}{U_0^2} - \frac{U^2}{U_0^2}\right), \tag{72}$$

where p_M represents the pressure at the matching point η_M, which is set at $6+4C$ in the present calculation. The value of $p_M - p_0$ is changed as a function of

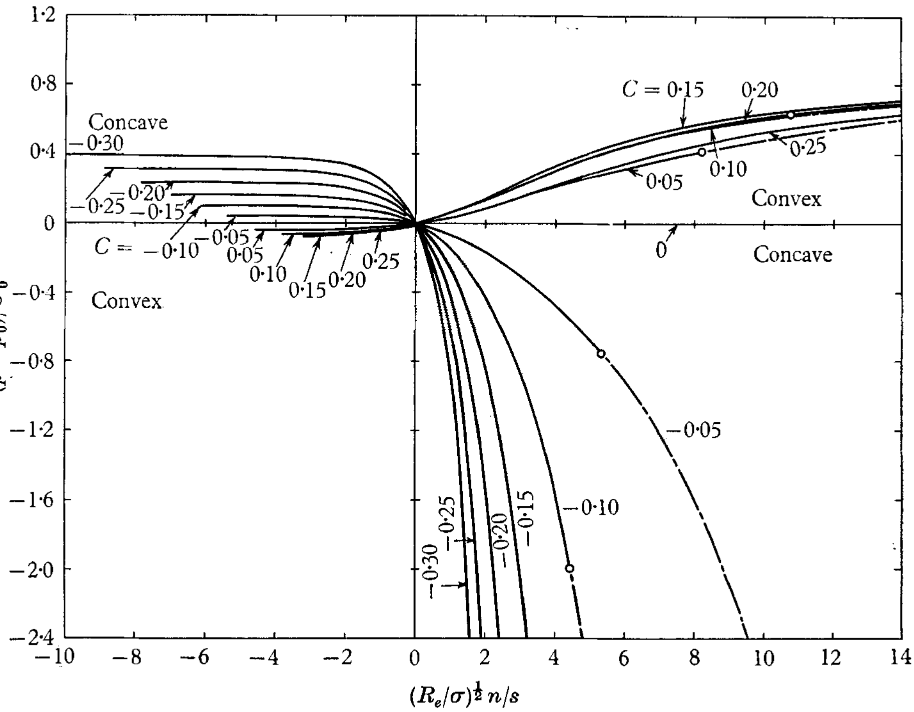

FIGURE 9. Pressure distributions, $(p-p_0)/U_0^2$.

C	$\dfrac{H_0^{-1}}{U_0}$	$\left(\dfrac{R_e}{\sigma}\right)^{\frac{1}{2}} \dfrac{(\tau_{\alpha\beta})_0}{U_0^2}$	$\dfrac{p_{-\infty}-p_0}{U_0^2}$	$\dfrac{p_0-p_M}{U_0^2}$	$\dfrac{p_M-p_\infty}{U_0^2}$ †
−0·30	0·9691	1·2182	0·3954	16·6888	∞
−0·25	0·9660	1·0507	0·3156	11·0606	∞
−0·20	0·9642	0·8943	0·2385	6·8929	∞
−0·15	0·9648	0·7502	0·1664	3·9591	∞
−0·10	0·9693	0·6193	0·1016	1·9957	∞
−0·05	0·9799	0·5022	0·0457	0·7486	∞
0	1·0000	0·3988	0	0	0
0·05	1·0351	0·3088	−0·0346	−0·4204	−0·5379
0·10	1·0953	0·2313	−0·0578	−0·6368	−0·2780
0·15	1·2011	0·1649	−0·0704	−0·7343	−0·1381
0·20	1·4034	0·1069	−0·0709	−0·7693	−0·0659
0·25	1·8818	0·0519	−0·0560	−0·7795	−0·0302

† Outer potential flow.

TABLE 1. Pressure difference and shearing stress (a)

η_M. If η_M tends to $+\infty$, the outer portion of flow approaches more and more that of potential flow, and therefore the value of $p_M - p$ will approach a constant

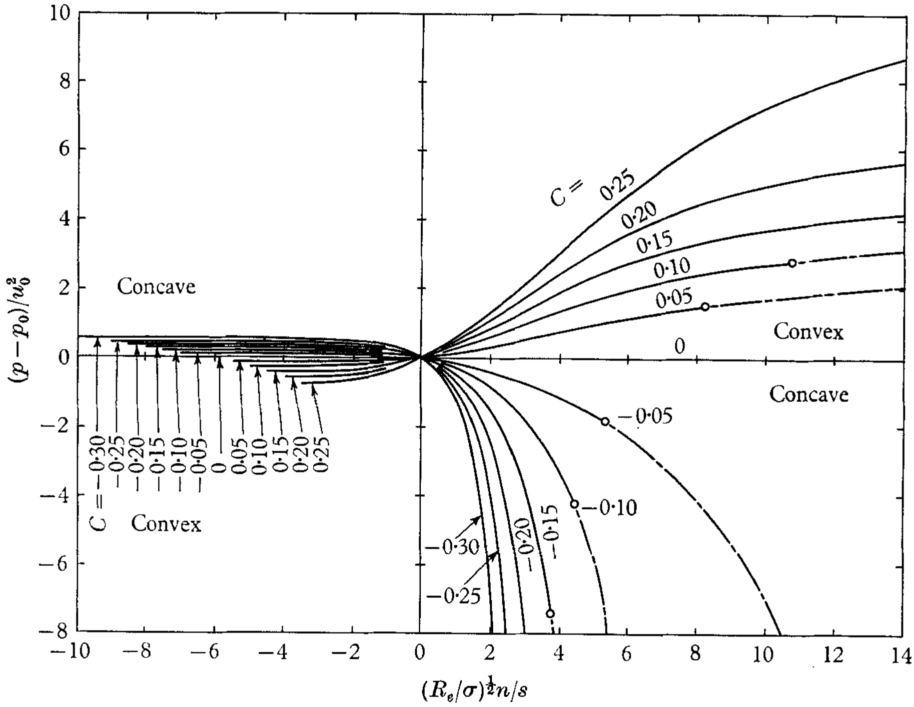

FIGURE 10. Pressure distributions, $(p - p_0)/u_0^2$.

C	$\dfrac{u_0}{U_0}$	$\left(\dfrac{R_e}{\sigma}\right)^{\frac{1}{2}}\dfrac{(\tau_{\alpha\beta})_0}{u_0^2}$	$\dfrac{p_{-\infty}-p_0}{u_0^2}$	$\dfrac{p_0-p_M}{u_0^2}$	$\dfrac{p_M-p_\infty}{u_0^2}$†
−0·30	0·8433	1·7131	0·5560	23·4696	∞
−0·25	0·8099	1·6017	0·4830	16·8609	∞
−0·20	0·7726	1·4980	0·3995	11·5463	∞
−0·15	0·7315	1·4018	0·3110	7·3979	∞
−0·10	0·6869	1·3127	0·2152	4·2299	∞
−0·05	0·6390	1·2300	0·1119	1·8335	∞
0	0·5881	1·1532	0	0	0
0·05	0·5344	1·0814	−0·1212	−1·4722	−1·8837
0·10	0·4779	1·0130	−0·2532	−2·7886	−1·2176
0·15	0·4177	0·9448	−0·4035	−4·2082	−0·7913
0·20	0·3516	0·8652	−0·5736	−6·2233	−0·5329
0·25	0·2720	0·7015	−0·7564	−10·5387	−0·4082

† Outer potential flow.

TABLE 2. Pressure difference and shearing stress (*b*)

value for a convex case and $-\infty$ for a concave case. Numerical examples are shown in figure 9 and in table 1.

In some cases the pressure distribution expressed in terms of the dynamic pressure on the zero-streamline of the mixing layer is conveniently used. It is simply calculated by dividing (71) and (72) by u_0^2/U_0^2. Some examples are shown in figure 10 and in table 2. It is found that for convex flows the pressure increases as C increases.

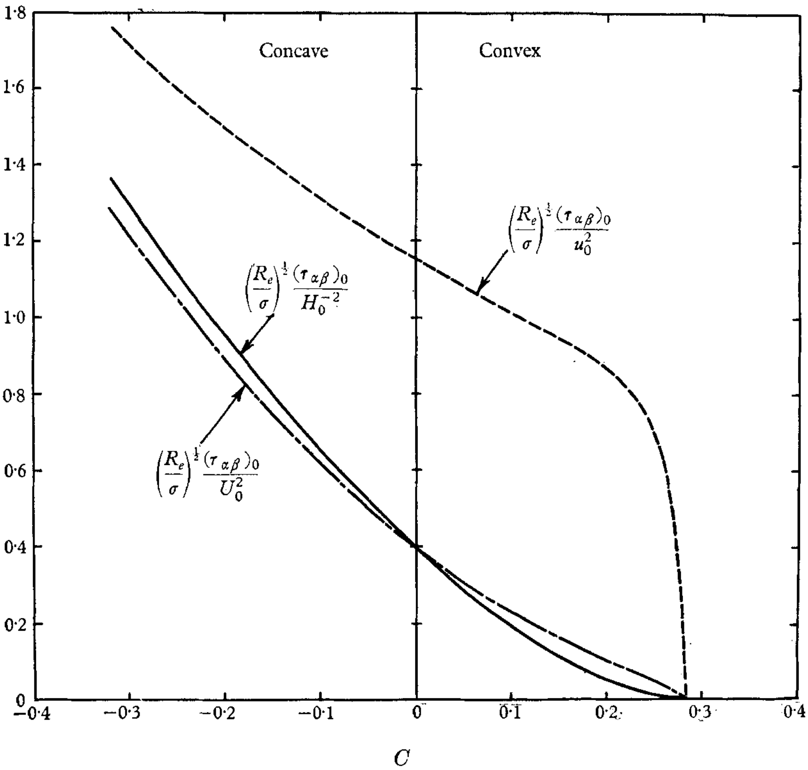

FIGURE 11. Shearing stress along zero-streamline.

The shearing stress along the zero-streamline is given by

$$(R_e/\sigma)^{\frac{1}{2}}\,(\tau_{\alpha\beta})_0/U_0^2 = 4(H_0^{-2}/U_0^2)\,[\Lambda(\Lambda'-2C)]_{\eta=0}. \tag{73}$$

The shearing stress expressed in terms of the dynamic pressure on the zero-streamline of mixing layer is given by

$$(R_e/\sigma)^{\frac{1}{2}}\,(\tau_{\alpha\beta})_0/u_0^2 = 4(H_0^{-2}/u_0^2)\,[\Lambda(\Lambda'-2C)]_{\eta=0}. \tag{74}$$

These values are calculated and are shown in figure 11 and in tables 1 and 2. With increasing the value of C, the shearing stress decreases until it reaches zero at $C = 0{\cdot}285$.

7. Conclusion

Curvature effects on the incompressible turbulent mixing in the free jet boundary are analysed theoretically. The influence of curvature and of turbulent mixing on the velocity and on the pressure distribution are investigated by a similar solution referred to the streamline co-ordinates. It is found that the effects of curvature are similar to those in laminar flows

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